

# Higher-Order Euclidean Sets

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# Example 1: E(7, 12)

Clough and Douthett 1991

Method for computing a maximally even set

$$E_\alpha(k_1, n_1) = \left\{ \left\lfloor \frac{in_1}{k_1} \right\rfloor : i = 0, 1, \dots, k_1 - 1 \right\}$$

n	k	i	calculate	round down
12	7	0	0	0
12	7	1	1.71428571	1
12	7	2	3.42857143	3
12	7	3	5.14285714	5
12	7	4	6.85714286	6
12	7	5	8.57142857	8
12	7	6	10.2857143	10

Toussaint 2005

Geometric method for computing a Euclidean set

Step 1: x x x x x x x . . . .

Step 2: x x x x x x x  
 . . . . .

Step 3: x x x x x  
 . . . . .  
 x x

Step 4: x x x  
 . . .  
 x x  
 x x  
 . .

Read down: [ 0 2 3 5 7 8 10 ]

-2 ↓

Transpose: [ 0 1 3 5 6 8 10 ]

# Example 2: E(7,12): Ableton Live

Top part:  
[0 1 3 5 6 8 10]

Bottom part:  
[0 2 3 5 7 8 10]

These two cycles are the same underlying structure, displaced by 2 time points.



The screenshot shows the Ableton Live interface for a MIDI clip named "1 Claves A 2". The clip is set to a 3/4 time signature and a key signature of one sharp (F#). The pattern is a Euclidean rhythm with 12 steps, 7 notes, and a 1/16 division. The notes are D1 and G0. The pattern is displayed in a piano roll view, showing the notes on a grid. The top part of the pattern is highlighted in yellow, and the bottom part is highlighted in blue. The interface also shows the transport controls, the mixer, and the clip's properties.

Transport: Link, Tap, 90.00, 3 / 4, 0%, 1 Bar, F# C Major, 2. 1. 1

Clip: 1 Claves A 2, All Ins: 1, All Channels: 0, In: Auto, Off, Main: -22.5

Clip Properties: Start: 1. 1. 1, End: 1. 4. 1, Duplicate, Loop, Position: 1. 1. 1, Length: 0. 3. 0, Signature: 4 / 4, Groove: Swing 16ths, Scale: F# C Major

Pattern: Euclidean, Rotation: 0, 2, 0, 0, Steps: 12, Density: 7, Division: 1/16, Auto, Generate

Piano Roll: D1, G0, Velocity: Randomize 100, Ramp: 100, 127, Deviation: 0

# Example 3: Clave – rotate first algorithm

## Hyper Euclidean Set Generator Rotate first method

### Global Controls

tempo span (n)

▶ 120 ▶ 16

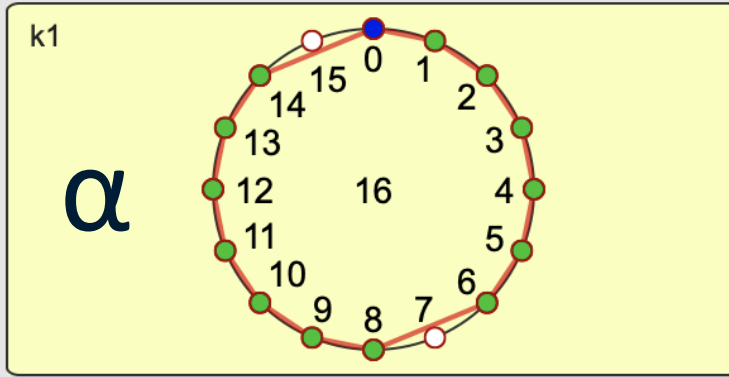
clock numbers on/off

pre-sets:

6 **clave son**

onsets (k1) rotate (k1)

▶ 14 ▶ 0  $E_1(14^0), 16$



note ▶ C4

onsets 0 1 2 3 4 5 6 8 9 10 11 12 13 14 IOI 1 1 1 1 1 2 1 1 1 1 1 1 2

play stop

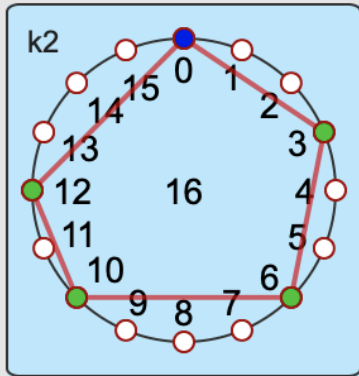
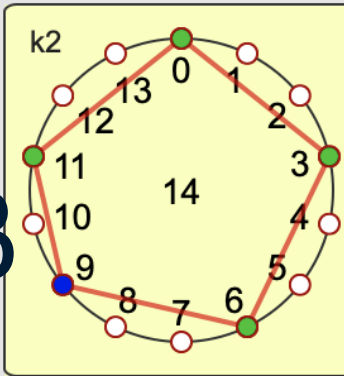
stdev 0.349927



onsets (k2) rotate (k2)

▶ 5 ▶ 9  $E_2(14^0, 5^9), 16$

$E_1(5^9), 14$



note ▶ D4

onsets 0 3 6 10 12 IOI 3 3 4 2 4

play stop

stdev 0.748331



# Example 4: Clave – rotate first algorithm

$$266: \text{HE} ( 16 14^0 5^9 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$394: \text{HE} ( 16 14^8 5^9 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$569: \text{HE} ( 16 13^3 5^8 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$649: \text{HE} ( 16 13^8 5^8 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$676: \text{HE} ( 16 13^{10} 5^3 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$756: \text{HE} ( 16 13^{15} 5^3 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$801: \text{HE} ( 16 12^2 5^0 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$813: \text{HE} ( 16 12^2 5^{12} ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$865: \text{HE} ( 16 12^6 5^0 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$877: \text{HE} ( 16 12^6 5^{12} ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$929: \text{HE} ( 16 12^{10} 5^0 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$941: \text{HE} ( 16 12^{10} 5^{12} ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$993: \text{HE} ( 16 12^{14} 5^0 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$1005: \text{HE} ( 16 12^{14} 5^{12} ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

# Example 5: Clave – subtract first algorithm

## Hyper Euclidean Set Generator

Subtract first method

### Global Controls

tempo

▶ 120

span (n)

▶ 16

clock numbers on/off

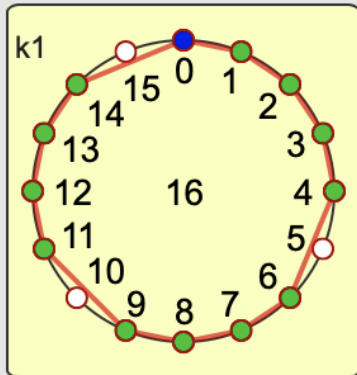
pre-sets:

6

clave son

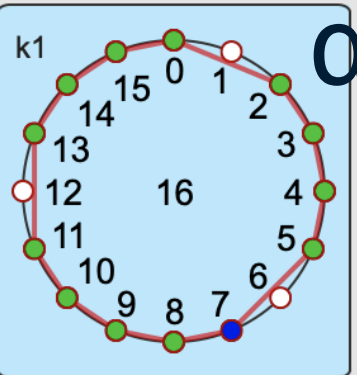
onsets (k1)

▶ 13  $E_1(13^0), 16$



rotate (r1)

▶ 7  $E_1(13^7), 16$



$\alpha$

note ▶ C4

onsets 0 2 3 4 5 7 8 9 10 11 13 14 15

IOI 2 1 1 1 2 1 1 1 1 2 1 1 1

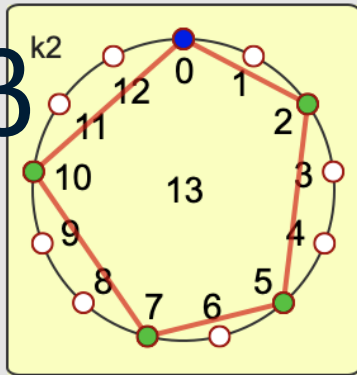
play stop

stdev 0.421325



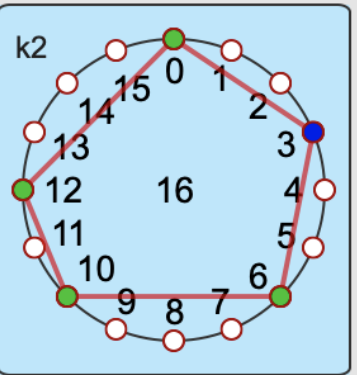
onsets (k2)

▶ 5  $E_1(5^0), 13$



rotate (r2)  $E_2(13^7, 5^3), 16$

▶ 3



$\beta$

note ▶ G4

onsets 0 3 6 10 12

IOI 3 3 4 2 4

play stop

stdev 0.748331



# Example 6: Clave – subtract first algorithm

$$363: \text{HE} ( 16 14^6 5^{10} ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$491: \text{HE} ( 16 14^{14} 5^{10} ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$628: \text{HE} ( 16 13^7 5^3 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$667: \text{HE} ( 16 13^9 5^{10} ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$708: \text{HE} ( 16 13^{12} 5^3 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$747: \text{HE} ( 16 13^{14} 5^{10} ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$801: \text{HE} ( 16 12^2 5^0 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$865: \text{HE} ( 16 12^6 5^0 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$929: \text{HE} ( 16 12^{10} 5^0 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

$$993: \text{HE} ( 16 12^{14} 5^0 ) = [ 0 3 6 10 12 ] < 3 3 4 2 4 >$$

# Example 7. A “Slate” for clave timeline calculation of depth 2 ( $p = 2$ )

	$n$	$k_1$	$k_2$	<i>number of sets</i>
Row 0:	16	15	5	$16 \times 16 = 256$
Row 1:	16	14	5	256
Row 2:	16	13	5	256
Row 3:	16	12	5	256
Row 4:	16	11	5	256
Row 5:	16	10	5	256
Row 6:	16	9	5	256
Row 7:	16	8	5	256
Row 8:	16	7	5	256
Row 9:	16	6	5	256

====

sum: 2560

Unfortunately, complexity increases at  $O(n^p)$ .





# Example 9. Deriving a fifth-order Euclidean set from the Clapping Music timeline

## Hyper Euclidean Set Generator

Subtract first method

**Global Controls**

tempo: 120 span (n): 12

clock numbers on/off:

pre-sets:

1 Clap Music (a)

Order	Onsets (k)	Rotate (r)	Note	Onsets	IOI	stdev
1	11	2	C4	0 2 3 4 5 6 7 8 9 10 11	2 1 1 1 1 1 1 1 1 1 1	0.28748
2	8	7	G4	0 1 2 4 5 7 9 10	1 1 2 1 2 2 1 2	0.5
3	7	0	A#4	0 1 2 4 5 7 9	1 1 2 1 2 2 3	0.699854
4	6	0	D5	0 1 2 4 5 7	1 1 2 1 2 5	1.414214
5	3	0	F#5	0 2 5	2 3 7	2.160247

# Example 10. "New Clapping Music"

## New Clapping Music

Paul V. Miller (2024)

Part 1: 12/8, whole rest.

Part 2: 12/8, whole rest, then rhythmic pattern starting at measure 4. Dynamics: *mf*.

Part 3: 12/8, rhythmic pattern starting at measure 1. Dynamics: *mf*.

Part 4: 12/8, rhythmic pattern starting at measure 1. Dynamics: *mf*.

Measures 4-7: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Measures 7-10: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Measures 10-13: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Measures 13-16: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Measures 16-19: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Measures 19-22: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Measures 10-13: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics. A box labeled "Repeat 4x" is around measures 10-13.

Measures 13-16: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Measures 16-19: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Measures 19-22: Part 1 has a *f* dynamic. Parts 2, 3, and 4 have *mf* dynamics.

Example 11. An example of a composition using higher-order Euclidean rhythms ( $105^0$ ,  $71^3$ ,  $48^9$ ,  $43^{11}$ ,  $12^{20}$ ), 128



A musical score for five parts in 4/4 time. The score is divided into two systems. The first system contains measures 1 through 6, and the second system contains measures 7 through 12. Each part is labeled 'part 1' through 'part 5'. The notation uses stems with flags to represent eighth notes and beams to represent sixteenth notes. Measure 12 is highlighted with an orange box, and an arrow points to it with the word 'empty' written below.

This slide intentionally left blank.

# Appendix 1: Producing higher-order sets

The maximally-even algorithm (Clough and Douthett 1991) produces a conventional first-order Euclidean set  $E_\alpha$  where  $n$  is the set's span, and  $k$  represents the number of desired onsets:

$$E_\alpha(k_1, n_1) = \left\{ \left\lfloor \frac{in_1}{k_1} \right\rfloor : i = 0, 1, \dots, k_1 - 1 \right\}$$

$E_\alpha$  is a set of onsets where  $j$  is the index of each  $e$ :

$$e_{\alpha_j} \in E_\alpha : j = 1, 2, \dots, k_1$$

To produce a **second-order Euclidean set**  $E_2$ , let  $n_2 = k_1$ , and  $k_2$  equal the number of desired onsets in the second iteration. Then by simple recursion,

$$E_\beta(k_2, n_2) = \left\{ \left\lfloor \frac{in_2}{k_2} \right\rfloor : i = 0, 1, \dots, k_2 - 1 \right\}$$

produces an **index set** such that each  $e_{\beta_j} \in E_\beta$  is the  $j$ th term in  $E_\alpha$ . The set of  $e_\beta$  onsets in  $E_\alpha$  is the second-order Euclidean set  $E_2$ . Consequently,  $E_2$  is *not* necessarily a subset of  $E_\alpha$ .



# Appendix 3. The harmonic minor scale as a second-order Euclidean set

## Hyper Euclidean Set Generator

Subtract first method

### Global Controls

tempo

▶ 120

span (n)

▶ 12

clock numbers on/off

pre-sets  
:

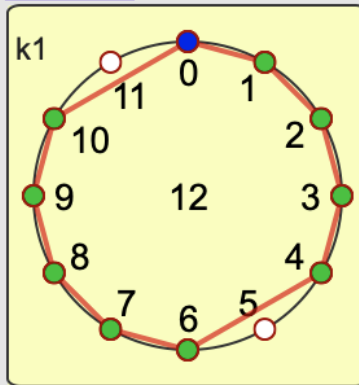


▶ 14

harm min scale

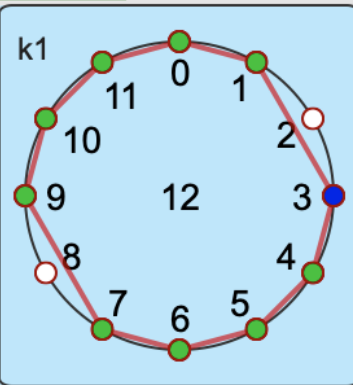
onsets (k1)

▶ 10



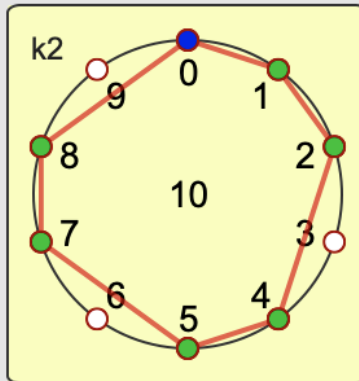
rotate (r1)

▶ 3



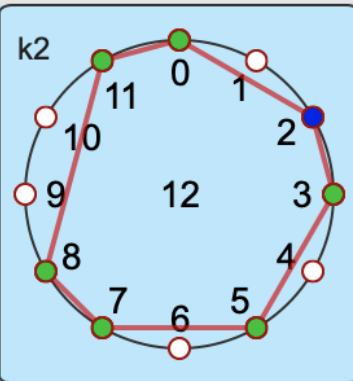
onsets (k2)

▶ 7



rotate (r2)

▶ 2



note ▶ C4

onsets 0 1 3 4 5 6 7 9 10 11

IOI 1 2 1 1 1 1 2 1 1 1

play stop

stdev 0.4



note ▶ G4

onsets 0 2 3 5 7 8 11

IOI 2 1 2 2 1 3 1

play stop

stdev 0.699854

